

# Large Deviations on a Cayley Tree I: Rate Functions.

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**Abstract.** We study the spherical model of a ferromagnet on a Cayley tree and show that in the case of empty boundary conditions the ferromagnetic phase transition takes place at the critical temperature  $T_c = \frac{6\sqrt{2}}{5}J$ , where  $J$  is the interaction strength. For any temperature the equilibrium magnetization,  $m_n$ , tends to zero in the thermodynamic limit, and the true order parameter is the renormalized magnetization  $r_n = n^{3/2}m_n$ , where  $n$  is the number of generations in the Cayley tree. Below  $T_c$ , the equilibrium values of the order parameter are given by

$$\rho^* = \pm \frac{2\pi}{(\sqrt{2}-1)^2} \sqrt{1 - \frac{T}{T_c}}.$$

There is one more notable temperature,  $T_p$ , in the model. Below that temperature the influence of homogeneous boundary field penetrates throughout the tree. We call  $T_p$  the penetration temperature, and it is given by

$$T_p = \frac{J}{W_{\text{Cayley}}(3/2)} \left( 1 - \frac{1}{\sqrt{2}} \left( \frac{h}{2J} \right)^2 \right).$$

The main new technical result of the paper is a complete set of orthonormal eigenvectors for the discrete Laplace operator on a Cayley tree.

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KEY WORDS: Critical temperature; order parameter; phase transition; spherical model

## 1 Introduction.

It is well known that some thermodynamic observables of the Ising model on a Cayley tree (the IC model) are non-analytic functions of the temperature. In the case of a tree with branching ratio  $k$ , the expected values of microscopic variables (local magnetization) are singular at  $T_B \equiv \beta_B^{-1}$ :  $\tanh(J\beta_B) = 1/k$ . However, the susceptibility of total magnetization diverges for  $T \leq T_C \equiv \beta_C^{-1}$ :  $\tanh(J\beta_C) = 1/\sqrt{k}$ , see [9] and [10]. The very existence of two ferromagnetic critical points is rather puzzling, because in finite-dimensional systems diverging susceptibility is usually

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accompanied by steeply rising spontaneous magnetization. The present paper is an attempt to shed some light on the mystery of this double critical-point phenomenon.

One of the distinctive features of Cayley trees is abnormally large number of boundary sites. In fact, the boundary is a macroscopic part of entire tree. Therefore, it does not come as a surprise, that properties of Cayley-tree models are most intriguing in the case of zero boundary field (the empty boundary conditions), see, e.g., [5, 9]. For example, below the critical temperature the distribution of total magnetization in the 2D Ising model or in the 3D spherical model with empty boundary conditions concentrates around two points  $\pm m^*(T)$ , where  $m^*(T)$  is the spontaneous magnetization. On the contrary, the distribution of magnetization in the IC model always concentrates around zero. That is, the total spontaneous magnetization in the IC model is equal to zero on the entire interval  $T < T_C$  where the susceptibility is infinite.

In order to resolve a paradoxical case or a seemingly contradictory situation it is important to present all available facts in a most clear and transparent way. To obtain such a description of important thermodynamic properties of a model (with or) without symmetry-breaking perturbations one can look at large-deviation probabilities of thermodynamic observables

$$\Pr[m_N \in [a, b]] \sim \exp \left[ -N \min_{x \in [a, b]} R(x) \right].$$

The typical behavior of rate functions  $R(x)$  within many models of statistical mechanics can be summarized by the following standard scenario. If the temperature is sufficiently high, the function  $R(x)$  has a positive second derivative and a unique minimum (in fact, zero) at the equilibrium value  $m(T)$  of the variable  $m_N$  under consideration. However, when the temperature drops to a critical value,  $T_c$ , the second derivative at the point of minimum vanishes,  $R''(m(T_c)) = 0$ . If the temperature is reduced even further, then the minimum of  $R(x)$  either stretches into a flat horizontal segment or, in the case of mean-field models, splits into several points of minima. In the former case the low-temperature phases are called soft, in the later case the phases are called rigid.

Another surprising (in view of the above standard scenario) property of Cayley-tree models was reported in the paper [4], where it was shown that the second derivative of the rate function  $R''_{IC}(x)$ , describing large-deviation probabilities of magnetization, does not vanish neither at nor below the critical point. The second derivative of a rate function at the minimum point determines the variance of fluctuations of the variable under consideration around its equilibrium value. When the second derivative tends to zero, the variance of fluctuations tends to infinity signaling that the thermodynamic system is approaching a critical point. Therefore, non-vanishing second derivative of rate functions in Cayley-tree models raises a question on the nature of phase transitions there.

The IC model has a close relative — the Ising model on a Bethe lattice (the IB model). A derivation of the exact formula for the magnetization of IB model can be found, for instance, in Section 4 of the famous book by Baxter [1]. What was also

derived in [1] is the free energy of IB model in the ensemble with fixed magnetization, whence the rate function of magnetization,  $R_{\text{IB}}(x)$ , can be extracted. Although the status of thus derived expressions is not quite clear, the obtained rate function  $R_{\text{IB}}(x)$  also exhibits a surprising feature established rigorously in [4] — the second derivative of  $R_{\text{IB}}(x)$  is strictly positive for all temperatures. Moreover,  $R''_{\text{IB}}(x)$  at the equilibrium value  $x = 0$  becomes a linear function of  $\beta J$ ,  $R''_{\text{IB}}(0) = \frac{1}{9}\beta J$ , for  $\beta > \beta_{\text{B}}$ .

From a geometric point of view, a Cayley tree is a kind of a loose bundle of 1D lattices. Can such a bundle exhibit a phase transition and support true criticality associated with emerging long-range correlations between macroscopically separated domains? It is quite conceivable that the singularity in the IB model has nothing to do with strong correlations. Instead, the critical point might signal penetration inside the tree of an effective field induced by boundary conditions imposed on a macroscopically large part of the lattice and its accumulation in a mesoscopic domain inside the tree. However, as far as some other models on Cayley trees are concerned, an evidence of strong correlations was established in [9], where it was shown that the susceptibility of the IC model diverges at  $T = T_{\text{C}}$ , and in [2], where it was shown that the free boson gas on a Cayley tree exhibits condensation in the ground state for sufficiently low temperatures (although, the exact value of the critical temperature was not reported in [2]).

Taking into account a host of intriguing features found in Cayley-tree models, it seems worthwhile to investigate the properties of the corresponding spherical model. First of all, it is interesting to find out which of the features found in models with discrete microscopic variables are also present in continuous models. Second, owing to the gaussian distribution of the microscopic random variables (above the critical temperature, in any case), one might hope to obtain a more complete and explicit description of thermodynamic properties of the spherical model, than the results obtained for the IC model so far. It turns out, that properties of the spherical and the Ising models on a Cayley tree are quite similar. In particular, the second derivative of the rate function  $R_{\text{Sph}}(x)$  describing large-deviation probabilities of the magnetization in the spherical model is also positive at the unique minimum  $x = 0$  for any temperature. Moreover, analogously to  $R''_{\text{IB}}(0)$ , the second derivative of  $R_{\text{Sph}}(x)$  at  $x = 0$  also simplifies to a linear function of  $\beta J$  for  $\beta > \beta_{\text{c}}$ :

$$R''_{\text{Sph}}(0) = \frac{(\sqrt{2} - 1)^2}{\sqrt{2}} \beta J.$$

Most likely, the behavior of rate functions in Cayley-tree models follows the mean-field scenario if we look at large-deviation probabilities of appropriate order parameters. In the case of the spherical model, the behaviour of the rate function  $R_{\text{Cayley}}(x)$  describing large-deviation probabilities of the renormalized magnetization,

$$r_{N, \frac{3}{2}} = (\log_2 N)^{\frac{3}{2}} m_N \equiv \frac{(\log_2 N)^{\frac{3}{2}}}{N} \sum_{(j,k) \in T_n} x_{j,k},$$

does follow the mean-field scenario. Namely, the unique minimum of  $R_{\text{Cayley}}(x)$  at  $x = 0$  splits in two isolated points of minima  $x = \pm \rho^*$ , and the rate function is no

longer convex when the temperature falls below  $T_c = \frac{6\sqrt{2}}{5}J$ . Nevertheless, the rate function  $R_{\text{Cayley}}(x)$  is also not devoid of unusual properties. As the temperature drops to its critical value  $T_c$ , the second derivative  $R''_{\text{Cayley}}(0)$  does not vanish, but tends to the positive value  $\frac{5}{12}(\sqrt{2} - 1)^2$ .

The rest of the paper is organized as follows. In Section 2 we define the spherical model on a Cayley tree and present several technical results that are used in the later sections. The main results of the paper are summarized in Section 3. In Section 4 we derive the main asymptotics of the free energy and establish the phase diagrams of the spherical model with three boundary conditions: empty, homogeneous, and alternating (antiferromagnetic). Section 5 is devoted to an investigation of large-deviation probabilities for magnetization. The ground-state (zero-temperature) properties of the spherical model are investigated in Section 6. In Section 7 we look at the large-deviation probabilities of the renormalized magnetization  $r_{N, \frac{3}{2}}$  — the true order parameter of the spherical model on a Cayley tree. The results of the paper are discussed in Section 8.

## 2 The model and useful facts.

Consider a binary Cayley tree  $T_n$  — a tree with branching ratio two containing  $n$  generations. Each node of the tree is labelled by a pair of integers  $(k, l)$ , where the first integer indicates the tree generation,  $k = 1, 2, \dots, n$ , and the second integer numbers nodes within the  $k$ -th generation,  $l = 1, 2, \dots, 2^{k-1}$ , see Fig. 1. There are exactly  $N = 2^n - 1$  nodes in a tree with  $n$  generations. The spherical model on a Cayley tree describes a collection of random variables  $\{x_{j,k} : (j, k) \in T_n\}$  attached to the nodes of the tree  $T_n$ .

### The Hamiltonian

The interaction of the variables  $x_{j,k}$  is described by the Hamiltonian

$$H_n = -J \sum_{(j,k),(l,m) \in T_n} M_{(j,k),(l,m)} x_{j,k} x_{l,m} - \sum_{k=1}^{2^{n-1}} h_k x_{n,k}, \quad (1)$$

where  $J > 0$ ,

$$M_{(j,k),(l,m)} = \begin{cases} \frac{1}{2}, & \text{if } l = j + 1, m \in \{2k - 1, 2k\}, \\ \frac{1}{2}, & \text{if } l = j - 1, k \in \{2m - 1, 2m\}, \\ 0, & \text{otherwise,} \end{cases} \quad (2)$$

are elements of the symmetric (nearest neighbour) Cayley-tree matrix  $\widehat{M}_N$ , and  $\{h_k : k = 1, 2, \dots, 2^{n-1}\}$  is a boundary field. Note that according to the Hamiltonian  $H_n$  only variables  $x_{j,k}$  located at nearest-neighbour nodes of the Cayley tree interact with each other directly. In this paper we consider three types of boundary conditions:  $h_k = 0$  (empty b.c.),  $h_k = h$  (homogeneous b.c.), and  $h_k = (-1)^k h$ , for  $k = 1, 2, \dots, 2^{n-1}$  (alternating b.c.).

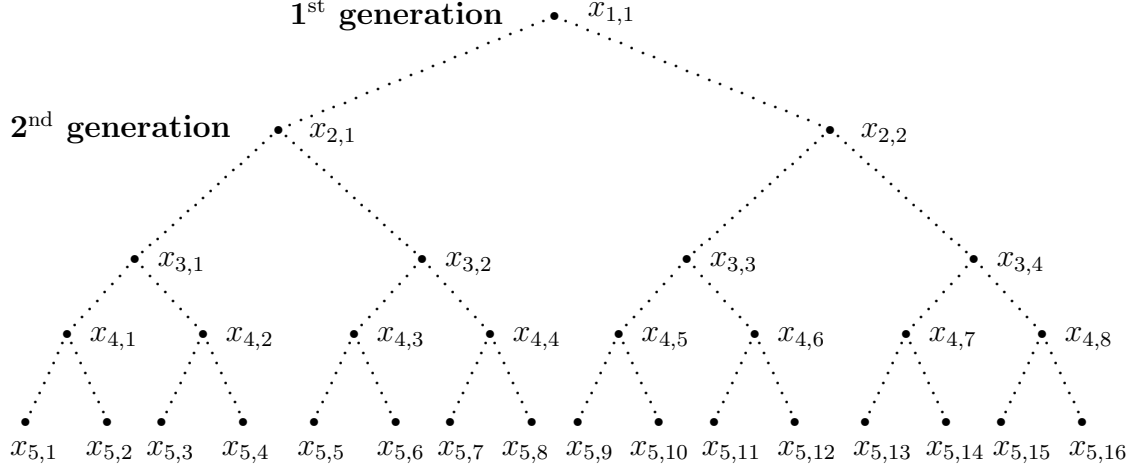


Figure 1: A binary Cayley tree  $T_5$  — a tree with branching ratio two containing five generations. A random variable  $x_{j,k}$  is attached to each node  $(j, k)$  of the tree. Nearest-neighbour nodes are connected by dotted lines.

The  $N$  eigenvalues,  $\{\lambda_{j,k} : (j, k) \in T_n\}$ , of the matrix  $\widehat{M}_N$  and the corresponding eigenvectors  $\{\mathbf{u}^{(j,k)} : (j, k) \in T_n\}$  are found in Appendix A. The spectrum of the matrix  $\widehat{M}_N$  (the set of different eigenvalues),  $\{\tau_{j,k}\}_{k=1, j=1}^j$ , contains exactly  $n(n+1)/2$  real numbers. It is convenient to arrange the values  $\tau_{j,k}$  and their multiplicities  $m_{j,k}$  in triangular arrays

$$\begin{array}{ll}
 \tau_{1,1}(=0) & m_{1,1} = 2^{n-2} \\
 \tau_{2,1}, \tau_{2,2} & m_{2,1} = m_{2,2} = 2^{n-3} \\
 \tau_{3,1}, \tau_{3,2}, \tau_{3,3} & m_{3,1} = m_{3,2} = m_{3,3} = 2^{n-4} \\
 \vdots & \vdots \\
 \tau_{n-1,1}, \tau_{n-1,2}, \dots, \tau_{n-1,n-1} & m_{n-1,1} = m_{n-1,2} = \dots = m_{n-1,n-1} = 1 \\
 \tau_{n,1}, \tau_{n,2}, \tau_{n,3}, \dots, \tau_{n,n-1}, \tau_{n,n} & m_{n,1} = m_{n,2} = m_{n,3} = \dots = m_{n,n} = 1
 \end{array} \tag{3}$$

In fact, the  $k$ -th line of the above triangular array contains the eigenvalues

$$\Lambda_{k;l} = \sqrt{2} \cos \frac{\pi l}{k+1}, \quad l = 1, 2, \dots, k$$

of the  $k \times k$  tri-diagonal matrix

$$\widehat{L}_k = \begin{pmatrix} 0 & 1 & & & 0 \\ \frac{1}{2} & 0 & 1 & & \\ & \frac{1}{2} & 0 & \ddots & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & 0 & 1 \\ & & & & \frac{1}{2} & 0 & 1 \\ & & & & & \frac{1}{2} & 0 \end{pmatrix}.$$

The normalized eigenvectors  $\mathbf{v}^{(n,l)}$  corresponding to the last line of eigenvalues  $\tau_{n,l}$  in Eq. (3) are given by

$$\left\{ v_{k,m}^{(n,l)} \right\}_{(k,m) \in T_n} = \left\{ \frac{2^{1-k/2}}{\sqrt{n+1}} \sin \frac{\pi l k}{n+1} \right\}_{(l,m) \in T_n}. \quad (4)$$

Note that the components  $v_{k,m}^{(n,l)}$  do not depend on  $m$ , that is, they are identical within each generation of the tree. Below, we refer to these vectors  $\mathbf{v}^{(n,l)}$  as special eigenvectors.

### The Gibbs distribution

The joint distribution of the random variables  $\{x_{j,k} : (j,k) \in T_n\}$  is specified by the usual Gibbs density

$$p(\{x_{j,k} : (j,k) \in T_n\}) = \frac{e^{-\beta H_n}}{\Theta_n},$$

with respect to the spherical “*a priori*” measure

$$\mu_n(dx) = \delta \left( \sum_{(j,k) \in T_n} x_{j,k}^2 - N \right) \prod_{(j,k) \in T_n} dx_{j,k}.$$

The normalization factor (partition function)  $\Theta_n$  is given by

$$\Theta_n = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{-\beta H_n} \mu_n(dx). \quad (5)$$

### Useful sums and asymptotic expansions

Using the explicit expressions for the spectrum of the Cayley-tree matrix and the corresponding multiplicities, see Eq. (3), we obtain

$$\begin{aligned} L_n(z) &\equiv \frac{1}{N} \sum_{(j,k) \in T_n} \ln(z - \lambda_{j,k}) \\ &= \frac{1}{N} \sum_{j=1}^{n-1} \sum_{k=1}^j 2^{n-1-j} \ln \left( z - \sqrt{2} \cos \frac{\pi k}{j+1} \right) + \frac{1}{N} \sum_{k=1}^n \ln \left( z - \sqrt{2} \cos \frac{\pi k}{n+1} \right). \end{aligned} \quad (6)$$

The identity

$$\sum_{k=1}^j \ln \left( z - \sqrt{2} \cos \frac{\pi k}{j+1} \right) = (j+1) \ln \frac{x_+(z)}{\sqrt{2}} + \ln \frac{1 - x_-^{2(j+1)}(z)}{\sqrt{z^2 - 2}}, \quad (7)$$

where

$$x_{\pm}(z) = \frac{z \pm \sqrt{z^2 - 2}}{\sqrt{2}},$$

see, e.g., [11], allows one to get rid of the summations over  $k$ . As a consequence, for  $z > \sqrt{2}$  we obtain the following large- $n$  asymptotic expansion

$$\frac{1}{N} \sum_{(j,k) \in T_n} \ln(z - \lambda_{j,k}) = L(z) + O\left(\frac{\log_2 N}{N}\right), \quad (8)$$

where

$$L(z) = \frac{3}{2} \ln \frac{x_+(z)}{\sqrt{2}} - \frac{1}{4} \ln(z^2 - 2) + \sum_{j=2}^{\infty} 2^{-j} \ln[1 - x_-^{2j}(z)]. \quad (9)$$

The definition of  $L(z)$  requires clarifications when  $z \leq \sqrt{2}$ . In this case instead of Eqs. (8) and (9) one can use the following “accelerated” asymptotic expansion

$$\frac{1}{N} \sum'_{(j,k) \in T_n} \ln\left(\tau_{n-1,1} + \frac{\zeta}{n^3} - \lambda_{j,k}\right) = A_{n-1} + \frac{\zeta}{n^3} B_{n-1} + O(n^{-6}),$$

where the prime indicates that the sum over  $(j, k)$  does not include the term corresponding to the maximal eigenvalue  $\tau_{n,1}$  and

$$\begin{aligned} A_n &= -\frac{1}{2} \ln 2 + \sum_{j=2}^n 2^{-j} \ln \frac{\sin \frac{\pi j}{n+1}}{\sin \frac{\pi}{n+1}}, \\ B_n &= \frac{1}{\sin \frac{\pi}{n+1}} \sum_{j=2}^n 2^{-j} \left( \cot \frac{\pi}{n+1} - j \cot \frac{\pi j}{n+1} \right) = \frac{5}{3} + O(n^{-2}). \end{aligned} \quad (10)$$

Finally, if  $z = \tau_{n,1} + \frac{\zeta}{N} - \lambda_{j,k}$ , then the large- $n$  asymptotic expansion of  $L_n(z)$  is given by

$$\frac{1}{N} \sum_{(j,k) \in T_n} \ln\left(\tau_{n,1} + \frac{\zeta}{N} - \lambda_{j,k}\right) = C_n + \frac{1}{N} \ln \frac{\zeta}{N} + B_n \frac{\zeta}{N} + O(N^{-2}), \quad (11)$$

where

$$C_n = A_n + \frac{1}{N} \ln \frac{n+1}{\sin^2 \frac{\pi}{n+1}}.$$

The following sum (it can be calculated using, for instance, the “contour summation” technique, see [11]) will also prove useful

$$\frac{1}{n+1} \sum_{k=1}^n \frac{\sin \frac{\pi n k}{n+1} \sin \frac{\pi j k}{n+1}}{z - \sqrt{2} \cos \frac{\pi k}{n+1}} = \frac{1}{\sqrt{2}} \frac{x_+^j(z) - x_-^j(z)}{x_+^{n+1}(z) - x_-^{n+1}(z)}. \quad (12)$$

### 3 The main results.

The main theme of the present paper are thermodynamic properties of the spherical model on a Cayley tree with branching ratio 2. The important results can be stated as follows.

1. In the case of empty boundary conditions the spherical model transits into a ferromagnetic state at the critical temperature  $T_c = \frac{6\sqrt{2}}{5}J$ .
2. Since nearly half of Cayley-tree sites belong to the boundary, the value and the very existence of the critical temperature depends on the type of boundary conditions imposed. In the spherical model with the alternating boundary field  $h_{n,k} = (-1)^k h$ , the critical temperature exists only if  $|h| < 4J$ , and it is given by

$$T_c(h) = \left[ 1 - \left( \frac{h}{4J} \right)^2 \right] \frac{6\sqrt{2}}{5}J.$$

3. For any temperature  $T$ , the rate function  $R_{\text{Sph}}(x)$  describing large-deviation probabilities of the magnetization

$$\Pr \left[ \frac{1}{N} \sum_{(j,k) \in T_n} x_{j,k} \in [a, b] \right] \sim \exp \left[ -N \min_{x \in [a, b]} R_{\text{Sph}}(x) \right]$$

is a strictly convex analytic function vanishing only at  $x = 0$ . The second derivative  $R''_{\text{Sph}}(0)$  at the minimum point is always positive and

$$R''_{\text{Sph}}(0) = \frac{(\sqrt{2} - 1)^2}{\sqrt{2}} \beta J,$$

for  $T < T_c$ .

4. The true order parameter of the spherical model is the renormalized magnetization

$$r_{N, \frac{3}{2}} = (\log_2 N)^{\frac{3}{2}} m_N \equiv \frac{(\log_2 N)^{\frac{3}{2}}}{N} \sum_{(j,k) \in T_n} x_{j,k}.$$

Below the critical temperature  $T_c$  the equilibrium values of  $r_{N, \frac{3}{2}}$  — zeroes of the corresponding rate function  $R_{\text{Cayley}}(\rho)$  — are given by  $\rho = \pm \rho^*$ , where

$$\rho^* \equiv \frac{2\pi}{(\sqrt{2} - 1)^2} \sqrt{1 - \frac{\beta_c}{\beta}}.$$

The rate function  $R_{\text{Cayley}}(\rho)$  is not convex, therefore, the low-temperature phases of the spherical model on a Cayley-tree are of mean-field type and rigid, in the terminology of the paper [4].

5. Homogeneous boundary conditions generate an effective field that penetrates toward the tree root if  $T < T_p$ . The penetration temperature is an analogue of the critical temperature of the Ising model on a Bethe lattice and it is given by

$$T_p = \frac{J}{W_{\text{Cayley}}(3/2)} \left( 1 - \frac{1}{\sqrt{2}} \left( \frac{h}{2J} \right)^2 \right),$$

where  $W_{\text{Cayley}}(z)$  is define in Eq. (17).



## 4 The phase diagram.

The partition function  $\Theta_n$  of the spherical model on a Cayley tree can be calculated using the technique described in the famous paper by Berlin and Kac [3]. Here we outline only the main steps.

Introduction of new integration variables  $\{y_{l,m} : (l, m) \in T_n\}$  in Eq. (5) via

$$x_{j,k} = \sum_{(l,m) \in T_n} u_{j,k}^{(l,m)} y_{l,m}, \quad (j, k) \in T_n,$$

where  $\{u^{(l,m)} : (l, m) \in T_n\}$  are orthonormal eigenvectors of the Cayley-tree matrix  $\widehat{M}_N$ , diagonalises the Hamiltonian (1). Therefore, we obtain the following formula for the partition function

$$\Theta_n = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp \left[ \beta J \sum_{(l,m) \in T_n} \lambda_{l,m} y_{l,m}^2 + \beta \sum_{(l,m) \in T_n} \phi_{l,m} y_{l,m} \right] \mu_n(dy),$$

where  $\{\lambda_{l,m} : (l, m) \in T_n\}$  are the eigenvalues of the matrix  $\widehat{M}_N$ , see Eq. (3), and

$$\phi_{l,m} = \sum_{k=1}^{2^{n-1}} u_{n,k}^{(l,m)} h_k \quad (13)$$

are “scalar products” of the boundary field and the eigenvectors of the matrix  $\widehat{M}_N$ .

The integral representation for the delta function in the “*a priori*” measure,

$$\delta \left( \sum_{(j,k) \in T_n} y_{j,k}^2 - N \right) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \exp \left[ s \left( N - \sum_{(j,k) \in T_n} y_{j,k}^2 \right) \right],$$

allows one to perform integration over the new variables  $\{y_{j,k} : (j, k) \in T_n\}$ . However, one can switch the order of integration over the variables  $y_{j,k}$  and  $s$  only after a shift of the integration contour for  $s$  to the right. The shift must assure that the real part of the quadratic form involving the variables  $y_{j,k}$  is negatively defined. Switching the integration order, integrating over  $\{y_{j,k} : (j, k) \in T_n\}$ , and introducing a new integration variable  $z$  via  $s = \beta J z$  we obtain

$$\Theta_n = \frac{\beta J}{2\pi i} \left( \frac{\pi}{\beta J} \right)^{N/2} \int_{-i\infty+c}^{+i\infty+c} dz \exp [N\beta\Phi_n(z)], \quad (14)$$

where

$$\Phi_n(z) = Jz - \frac{1}{2\beta N} \sum_{(j,k) \in T_n} \ln(z - \lambda_{j,k}) + \frac{1}{4JN} \sum_{(j,k) \in T_n} \frac{\phi_{j,k}^2}{z - \lambda_{j,k}},$$

and  $c > \sqrt{2}$  is the shift of integration contour mentioned above.

The large- $n$  asymptotics of the integral (14) can be found using the saddle-point method. In the case of empty boundary conditions all scalar products  $\phi_{j,k}$  are equal to zero, and the saddle point of the integrand is a solution of the equation

$$\Phi'_n(z) = 0 \quad \Leftrightarrow \quad J - \frac{1}{2\beta N} \sum_{(j,k) \in T_n} \frac{1}{z - \lambda_{j,k}} = 0. \quad (15)$$

Making use of the explicit expressions for the spectrum  $\tau_{j,k}$  of the matrix  $\widehat{M}_N$  and the corresponding multiplicities  $m_{j,k}$ , see Eq. (3), we obtain

$$J - \frac{1}{2\beta N} \left[ \sum_{j=1}^{n-1} 2^{n-1-j} \sum_{k=1}^j \frac{1}{z - \sqrt{2} \cos \frac{\pi k}{j+1}} + \sum_{k=1}^n \frac{1}{z - \sqrt{2} \cos \frac{\pi k}{n+1}} \right] = 0.$$

Differentiation of Eq. (7) over  $z$  yields

$$\sum_{k=1}^j \frac{1}{z - \sqrt{2} \cos \frac{\pi k}{j+1}} = \frac{j+1}{\sqrt{z^2 - 2}} \left( 1 + \frac{2}{x_+^{2(j+1)}(z) - 1} \right) - \frac{z}{z^2 - 2}.$$

Therefore, assuming  $z > \sqrt{2}$  and passing to the limit  $n \rightarrow \infty$  we obtain the following equation for the saddle-point  $z^*$ :

$$\Phi'(z) \equiv J - \frac{1}{2\beta} W_{\text{Cayley}}(z) = 0, \quad (16)$$

where

$$W_{\text{Cayley}}(z) = \sum_{j=2}^{\infty} 2^{-j} \left[ \frac{j}{\sqrt{z^2 - 2}} \left( 1 + \frac{2}{x_+^{2j}(z) - 1} \right) - \frac{z}{z^2 - 2} \right] \quad (17)$$

is the Cayley-tree analogue of the Watson function from the original paper on the spherical model by Berlin and Kac, see [3].

The function  $\Phi'(z)$  increases monotonically with  $z$  on  $(\sqrt{2}, \infty)$ , and the location of its zeroes depends on the value of the inverse temperature  $\beta$ . Since

$$\Phi'(\sqrt{2}) = \lim_{z \downarrow \sqrt{2}} \Phi'(z) = J - \frac{5}{6\sqrt{2}\beta},$$

there exists a critical value

$$\beta_c = \frac{5}{6\sqrt{2}J}$$

of the inverse temperature  $\beta$ . If  $\beta \in (0, \beta_c)$  (high-temperatures), then the function  $\Phi'(z)$  has exactly one zero on the interval  $(\sqrt{2}, \infty)$  at a point  $z^* > \sqrt{2}$ . While if  $\beta > \beta_c$  (low-temperatures), then the function  $\Phi'(z)$  is strictly positive for  $z > \sqrt{2}$ .

The alternating boundary field  $h_k = (-1)^k h$  is orthogonal to all eigenvectors of the Cayley-tree matrix  $\widehat{M}_N$  except for the  $2^{n-2}$  eigenvectors corresponding to the eigenvalue  $\tau_{1,1} = 0$ . A short calculation shows, that non-zero scalar products (13)

are given by  $\phi_{l,m} = \sqrt{2}h$ . Therefore, the saddle point of the integrand in Eq. (14) is a solution of the equation

$$\Phi'_n(z) = 0 \quad \Leftrightarrow \quad J - \frac{1}{2\beta N} \sum_{(j,k) \in T_n} \frac{1}{z - \lambda_{j,k}} - \frac{1}{4JN} \frac{2h^2 2^{n-2}}{z^2} = 0. \quad (18)$$

Assuming  $z > \sqrt{2}$  and passing to the limit  $n \rightarrow \infty$  we obtain the following equation for the saddle-point  $z^*$ :

$$\Phi'(z) \equiv J - \frac{1}{2\beta} W_{\text{Cayley}}(z) - \left(\frac{h}{4J}\right)^2 \frac{2}{z^2} = 0. \quad (19)$$

Thus, in the case of alternating boundary conditions the critical temperature exists only if  $|h| < 4J$ , and it is given by

$$T_c^{\text{alt}}(h) = \left[ 1 - \left(\frac{h}{4J}\right)^2 \right] \frac{6\sqrt{2}J}{5}.$$

If the boundary field is homogeneous,  $h_k = h$ , then its “scalar products” with the special eigenvectors  $\mathbf{v}^{(n,m)}$ , see Eq. (13), are given by

$$\phi_{n,m} = \frac{2^{n/2}h}{\sqrt{n+1}} \sin \frac{\pi mn}{n+1}, \quad m = 1, 2, \dots, n.$$

All other eigenvectors of the Cayley-tree matrix are orthogonal to a homogeneous boundary field. Therefore, Eq. (12) yields the following formula for the function  $\Phi_n(z)$  in the integral (14)

$$\Phi_n(z) = Jz - \frac{1}{2\beta N} \sum_{(j,k) \in T_n} \ln(z - \lambda_{j,k}) + \frac{2^n h^2}{4JN\sqrt{2}} \frac{x_+^{2n+1}(z) - x_+(z)}{x_+^{2n+2}(z) - 1}. \quad (20)$$

Assuming  $z > \sqrt{2}$ , differentiating over  $z$  and passing to the limit  $n \rightarrow \infty$  we obtain the following saddle-point equation

$$J - \frac{1}{2\beta} W_{\text{Cayley}}(z) - \frac{h^2}{8J} \frac{x_-(z)}{\sqrt{z^2 - 2}} = 0. \quad (21)$$

As could have been expected, any homogeneous boundary field  $h \neq 0$  keeps the saddle point  $z^*$  away from the singularity at  $\sqrt{2}$  and, hence, destroys the phase transition.

Application of the saddle-point method to the integral (14) is straightforward when the saddle point  $z^*$  is greater than  $\sqrt{2}$ , see [3]. In the case of alternating boundary conditions, taking into account the estimate (8) we obtain the following expression for the main asymptotics of the partition function

$$\Theta_n = \left(\frac{\pi}{\beta J}\right)^{N/2} \exp \left[ N \left( \beta J z^* - \frac{1}{2} L(z^*) + \frac{\beta h^2}{8J z^*} \right) + O(n) \right], \quad (22)$$

where  $L(z)$  is given by Eq. (9). Analogously, in the case of homogeneous boundary conditions we obtain

$$\Theta_n = \left( \frac{\pi}{\beta J} \right)^{N/2} \exp \left[ N \left( \beta J z^* - \frac{1}{2} L(z^*) + \frac{\beta h^2}{4J} \frac{1}{z^* + \sqrt{(z^*)^2 - 2}} \right) + O(n) \right], \quad (23)$$

where  $z^*$  is the solution of Eq. (21) from the interval  $(\sqrt{2}, \infty)$ .

If  $\beta \geq \beta_c^{\text{alt}}(h)$ , the function  $\Phi_n(z)$  still attains its minimum value on the interval  $(\tau_{n,1}, \infty)$  at a point  $z_n^* > \tau_{n,1}$ , where  $\tau_{n,1} \equiv \sqrt{2} \cos \frac{\pi}{n+1}$  is the largest eigenvalue of the Cayley-tree matrix  $\widehat{M}_N$ . However, now the sequence of saddle points  $z_n^*$  approaches the branch point of the integrand at  $z = \tau_{n,1}$ , and application of the saddle-point method becomes a bit more tricky. In fact, in the cases of empty and alternating boundary conditions with  $|h| < 4J$ , nothing prevents the saddle point in Eq. (14) from sliding towards the branch point, and to find the main asymptotics of the partition function  $\Theta_n$  we have to introduce a new integration variable  $\zeta$  via  $z = \tau_{n,1} + \zeta/N$ . This change of variables effectively eliminates the large parameter,  $N$ , in the  $\zeta$ -dependent part of the integrand, and we are left with a finite integral. Thus, the main asymptotics of the partition function is given by

$$\Theta_n = \left( \frac{\pi}{\beta J} \right)^{N/2} \exp \left[ N \left( \beta J \tau_{n,1} - \frac{1}{2} A_n + \frac{\beta h^2}{8J \tau_{n,1}} \right) + O(n) \right], \quad (24)$$

where  $A_n$  is given by Eq. (10).

## 5 Large Deviations of Magnetization.

The main objective of this section is investigation of large-deviation probabilities

$$\Pr \left( \frac{1}{N} \sum_{(j,k) \in T_n} x_{j,k} \in [a; b] \right)$$

for the magnetization — the arithmetic average of microscopic random variables  $x_{j,k}$ ,  $(j,k) \in T_n$ . In particular, we would like to find the main asymptotics of the distribution densities

$$f_n(y) = \frac{1}{\Theta_n} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \delta \left( \frac{1}{N} \sum_{(j,k) \in T_n} x_{j,k} - y \right) \exp[-\beta H_n] \mu_n(dx), \quad (25)$$

as  $N \rightarrow \infty$ .

One can calculate the density  $f_n(y)$  using the technique from the previous section. This time, however, we have to deal with one more delta function. Using the integral representation

$$\delta(a) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iaw} dw$$

for the new delta function, and performing the integration over the variables  $x_{j,k}$  one obtains

$$f_n(y) = \frac{N}{4\pi^2 i \Theta_n} \int_{-i\infty+c}^{i\infty+c} ds e^{Ns} \int_{-\infty}^{\infty} dw e^{iNyw} \times \\ \times \prod_{(j,k) \in T_n} \sqrt{\frac{\pi}{s - \beta J \lambda_{j,k}}} \exp \left[ -\frac{\gamma_{j,k}^2 w^2}{4(s - \beta J \lambda_{j,k})} \right], \quad (26)$$

where

$$\gamma_{j,k} \equiv \sum_{(l,m) \in T_n} u_{l,m}^{(j,k)}.$$

Integrating over the variable  $u$  and introducing a new integration variable  $z$  via  $s = \beta J z$  one arrives at

$$f_n(y) = \frac{\beta J N}{2\pi i \Theta_n} \left( \frac{\pi}{\beta J} \right)^{(N-1)/2} \int_{-i\infty+z_0}^{i\infty+z_0} \frac{dz}{\sqrt{\Sigma_n(z)}} \exp [N\beta J \Phi_n(z, y)], \quad (27)$$

where we have introduced the notations

$$\Sigma_n(z) = \sum_{(j,k) \in T_n} \frac{\gamma_{j,k}^2}{z - \lambda_{j,k}}, \quad (28)$$

$$\Phi_n(z, y) = z - \frac{1}{2\beta J N} \sum_{(j,k) \in T_n} \ln(z - \lambda_{j,k}) - N \frac{y^2}{\Sigma_n(z)}. \quad (29)$$

If  $z > \sqrt{2}$ , then using Eq. (40) from Appendix B we obtain the following limit

$$\Phi(z, y) \equiv \lim_{n \rightarrow \infty} \Phi_n(z, y) = z - \frac{1}{2\beta J} L(z) - y^2 \sigma(z), \quad (30)$$

where

$$\sigma(z) = \frac{2 \left( z - \frac{3}{2} \right)^2}{3z - \sqrt{z^2 - 2} - 4}.$$

The Taylor expansion for  $\sigma(z)$  at the point  $z = \sqrt{2}$  is given by

$$\sigma(z) = \frac{3}{2\sqrt{2}} - 1 + 2^{-5/4} \sqrt{z - \sqrt{2}} + \frac{3(z - \sqrt{2})}{4} + O \left[ (z - \sqrt{2})^{3/2} \right].$$

The term  $\sqrt{z - \sqrt{2}}$  does not allow the saddle-points  $z_n(y)$  of  $\Phi_n(z, y)$  to approach the singularity of the integrand at  $z = \sqrt{2}$ . Therefore, provided  $y \neq 0$ , application of the saddle-point method to the integral (27) is straightforward. For the main asymptotics of the distribution density we obtain

$$f_N(y) = e^{-NR_{\text{Sph}}(y) + O(\log_2 N)},$$

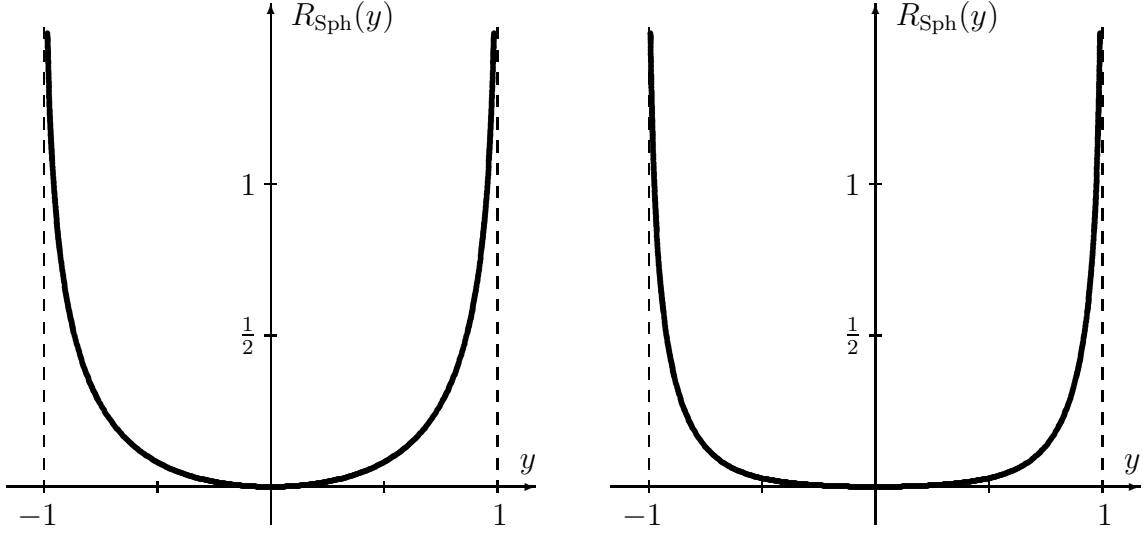


Figure 2: The rate functions for the magnetization of the spherical model on a Cayley tree above ( $\beta = \frac{1}{2}\beta_c$ , left) and below ( $\beta = 2\beta_c$ , right) the critical temperature. In both cases  $R_{\text{Sph}}(y) = 0$  only for  $y = 0$ , and  $R''_{\text{Sph}}(y) > 0$  for any  $y \in (-1, 1)$ .

where

$$R_{\text{Sph}}(y) = \beta J(z^* - z^*(y)) - \frac{1}{2}L(z^*) + \frac{1}{2}L(z^*(y)) + \beta Jy^2\sigma(z^*(y)), \quad (31)$$

is the rate function,  $z^*(y)$  is the minimum point of the function  $\Phi(z, y)$  on the interval  $[\sqrt{2}; \infty)$ , and  $z^* = z^*(0)$ .

The rate function  $R_{\text{Sph}}(y)$  has the following properties (see also Fig. 2):

1.  $R_{\text{Sph}}(y)$  is a non-negative, even, strictly convex function, and  $R_{\text{Sph}}(0) = 0$ ;
2.  $R_{\text{Sph}}(y) \sim -\frac{1}{2}\ln(1 - y^2)$ , as  $y \rightarrow \pm 1$ ;
3.  $R''(0) = \frac{3-2\sqrt{2}}{\sqrt{2}}\beta J$ , for  $\beta \geq \beta_c$ .

## 6 Ground State Properties.

It is always instructive to have a look at the ground-state (zero-temperature) properties of the model under investigation. In doing that we gain useful physical intuition and get an idea what one could expect to find for non-zero temperatures.

It is shown in Appendix A that the unit-length eigenvectors corresponding to the maximum eigenvalue of the Cayley-tree matrix are given by

$$\{v_{l,m}^{(n,1)}\}_{(l,m) \in T_n} = \pm \left\{ \frac{2^{1-l/2}}{\sqrt{n+1}} \sin \frac{\pi l}{n+1} \right\}_{(l,m) \in T_n}.$$

Since configurations of the spherical model obey the constraint

$$\sum_{(l,m) \in T_n} x_{l,m}^2 = N \equiv 2^n - 1,$$

the pair of ground-state configurations is given by

$$\mathbf{g}_{\pm} = \pm \left\{ \sqrt{\frac{2^{2-l}N}{n+1}} \sin \frac{\pi l}{n+1} \right\}_{(l,m) \in T_n}.$$

Now a simple calculation (see the calculation of  $\gamma_{n,k}$  in Appendix B) shows that the large- $n$  asymptotics of the ground-state magnetization is given by

$$m_n = \pm \frac{2\pi}{(\sqrt{2}-1)^2} n^{-3/2} + O(n^{-5/2}),$$

in the “+” and the “−” phase, respectively. Thus, even below the critical temperature we should not expect to obtain non-zero spontaneous magnetization in an infinite-tree limit. Instead we have to look at the renormalized magnetization

$$\rho_n = \frac{n^{3/2}}{N} \sum_{(l,m) \in T_n} x_{l,m},$$

which, as it turns out, is the true order parameter for the spherical model of a ferromagnet on a Cayley tree.

## 7 Large Deviations of the Order Parameter.

The results of zero-temperature analysis and the strict convexity of the rate function  $R_{\text{Sph}}(y)$  below the critical temperature, see Eq. (31) and Fig. 2, hint that our choice of the normalization for magnetization is not quite right. That is, although the distribution of

$$m_N = \frac{1}{N} \sum_{(j,k) \in T_n} x_{j,k}$$

is indeed asymptotically degenerate and concentrates at 0, as  $N \rightarrow \infty$ , but if we replace  $N$  by a softer normalization, then we could go back to the usual situation where the distribution of order parameter concentrates at two points  $\pm \rho^*$ . This is precisely the result that we are going to establish in the present section.

For the distribution density of the renormalized magnetization

$$r_{N,\gamma} = \frac{n^\gamma}{N} \sum_{(j,k) \in T_n} x_{j,k},$$

where  $\gamma > 0$ , we obtain

$$f_{N,\gamma}(\rho) = \frac{\beta J N}{2\pi i \Theta_N} \left( \frac{\pi}{\beta J} \right)^{(N-1)/2} \int_{-i\infty+z_0}^{i\infty+z_0} \frac{dz}{\sqrt{\Sigma_n(z)(z-\tau_{n,1})}} \exp [N\beta J \Gamma_n(z, \rho)], \quad (32)$$

where

$$\Gamma_n(z, \rho) = z - \frac{1}{2\beta JN} \sum'_{(j,k) \in T_n} \ln(z - \lambda_{j,k}) - \frac{N\rho^2}{n^{2\gamma}\Sigma_n(z)},$$

and the prime indicates that the sum over  $(j, k)$  does not include the term  $\ln(z - \tau_{n,1})$  corresponding to the (non-degenerate) maximal eigenvalue. The multiplier  $n^{2\gamma}$  suppresses the blocking influence of  $\Sigma_n(z)$  and allows the saddle-point of  $\Gamma_n(z, \rho)$  to enter the immediate vicinity of the eigenvalue  $\tau_{n,1}$  when  $\beta \geq \beta_c$ . But first we have to find the distribution of  $r_{N,\gamma}$  for high temperatures.

If  $\beta < \beta_c$ , then the relevant saddle points  $z_n(\rho)$  of  $\Gamma_n(z, \rho)$  do not approach  $\tau_{n,1}$  and evaluation of the integral (32) is straightforward. The relevant solution of the saddle-point equation

$$1 - \frac{L'(z)}{2\beta J} - \frac{\rho^2}{n^{2\gamma}} \sigma'(z) = 0$$

is given by

$$z_n(\rho) = z^* - 2\beta J \rho^2 \frac{\sigma'(z^*)}{L''(z^*)} n^{-2\gamma} + O(n^{-4\gamma}),$$

where  $z^*$  is the maximal solution of Eq. (16).

Therefore, for  $\beta < \beta_c$  the main asymptotics of the distribution density of  $r_{N,\gamma}$  is given by

$$f_{N,\gamma}(\rho) = \exp \left[ -\frac{N}{n^{2\gamma}} \beta J \sigma(z^*) \rho^2 + O(Nn^{-4\gamma}) \right]. \quad (33)$$

Note that the quadratic rate function  $\beta J \sigma(z^*) \rho^2$  can be obtained formally from the rate function  $R_{\text{Sph}}(y)$ , see Eq. (31), if we substitute  $n^{-\gamma} \rho$  instead of  $y$ . However, below the critical temperature the situation becomes very different.

To evaluate the integral in Eq. (32) for  $\beta > \beta_c$  we have to locate singularities of the integrand in the vicinity of the maximal eigenvalue  $z = \tau_{n,1}$ . The singularities of  $L_n(z)$  and  $\Sigma_n(z)$  at

$$\tau_{n,1} = \sqrt{2} \left( 1 - \frac{\pi^2}{2} n^{-2} + \pi^2 n^{-3} \right) + O(n^{-4})$$

cancel each other, therefore the integrand is analytic at  $z = \tau_{n,1}$ . The second-largest eigenvalue of the Cayley-tree matrix

$$\tau_{n-1,1} = \sqrt{2} \left( 1 - \frac{\pi^2}{2} n^{-2} \right) + O(n^{-4})$$

is also non-degenerate and the point  $z = \tau_{n-1,1}$  is the sticking point — the most important singularity of the integrand. It prevents the saddle-point from sliding further to the left when  $\rho^2$  is small, and the eigenvector  $\mathbf{v}^{(n-1,1)}$ , corresponding to  $\tau_{n-1,1}$ , is the state absorbing the macroscopic “condensation” taking place when the temperature drops below a certain critical level  $T_{\text{ph.s.}}$  where phase separation begins, see Eq. (36).



Another important singularity of the integrand is the maximal zero of  $\Sigma_n(z)$  at

$$s_n = \sqrt{2} \left( 1 - \frac{\pi^2}{2} n^{-2} - 2\pi^2(\sqrt{2} + 1)n^{-3} \right) + O(n^{-4}).$$

It however lies to the left of  $\tau_{n-1,1}$ , a bit further away from  $\tau_{n,1}$ , and, hence, in the present set up  $s_n$  is not the sticking point. That is,  $z = s_n$  is not the singularity preventing the saddle-point from sliding further to the left. The main reason for that is orthogonality of the eigenvector  $\mathbf{v}^{(n-1,1)}$  and the constant vector  $x_{j,k} = 1$  associated with the conventional and renormalized magnetizations,  $m_N$  and  $r_{N,\gamma}$ , respectively. However, if we decide to look at large-deviation probabilities of other observables, for instance, of the magnetization of a subdomain  $D_n \subset T_n$ , then the situation could become very different, and the maximal zero of  $\Sigma_n(z)$  could become the sticking point.

The formulae for  $\tau_{n,1}$ ,  $\tau_{n-1,1}$ , and  $s_n$  suggest that in order to evaluate the integral (32) by the saddle-point method we have to introduce a new integration variable  $\zeta$  via

$$z = \sqrt{2} \left( 1 - \frac{\pi^2}{2} n^{-2} + \pi^2 \zeta n^{-3} \right).$$

If  $z > \tau_{n-1,1}$ , that is, if  $\zeta > 0$ , then the large- $n$  asymptotic expansion of the function  $\Gamma_n(z, \rho)$  is given by

$$\begin{aligned} \Gamma_n \left[ \sqrt{2} \left( 1 - \frac{\pi^2}{2n^2} + \pi^2 \zeta n^{-3} \right), \rho \right] &= \sqrt{2} \left( 1 - \frac{\pi^2}{2n^2} + \pi^2 \zeta n^{-3} \right) \left( 1 - \frac{\beta_c}{\beta} \right) \\ &\quad - \frac{1}{n^{2\gamma} 2\sqrt{2}} \frac{(\sqrt{2} - 1)^2 (\zeta - 1) \rho^2}{(\sqrt{2} + 1)^2 + (\zeta - 1)} + O(n^{-2(1+\gamma)}). \end{aligned}$$

The two  $\zeta$ -dependent terms are of the same magnitude if  $\gamma = \frac{3}{2}$ . Differentiating the function  $\Gamma_n(z, \rho)$  (with  $\gamma$  replaced by  $\frac{3}{2}$ ) over  $\zeta$  we obtain the following equation for the saddle point  $\zeta^*$ :

$$\left( 1 - \frac{\beta_c}{\beta} \right) \pi^2 - \frac{\rho^2}{4} \frac{1}{((\sqrt{2} + 1)^2 + \zeta - 1)^2} = 0.$$

Since  $z = \tau_{n-1,1}$  is the sticking point, the relevant solution  $\zeta^*$  must be non-negative. Therefore

$$\zeta^* = -2(1 + \sqrt{2}) + \frac{|\rho|}{2\pi\sqrt{1 - \frac{\beta_c}{\beta}}},$$

if

$$|\rho| > \rho_c \equiv 4\pi(\sqrt{2} + 1)\sqrt{1 - \frac{\beta_c}{\beta}}, \quad (34)$$

and  $\zeta^* = 0$ , otherwise.

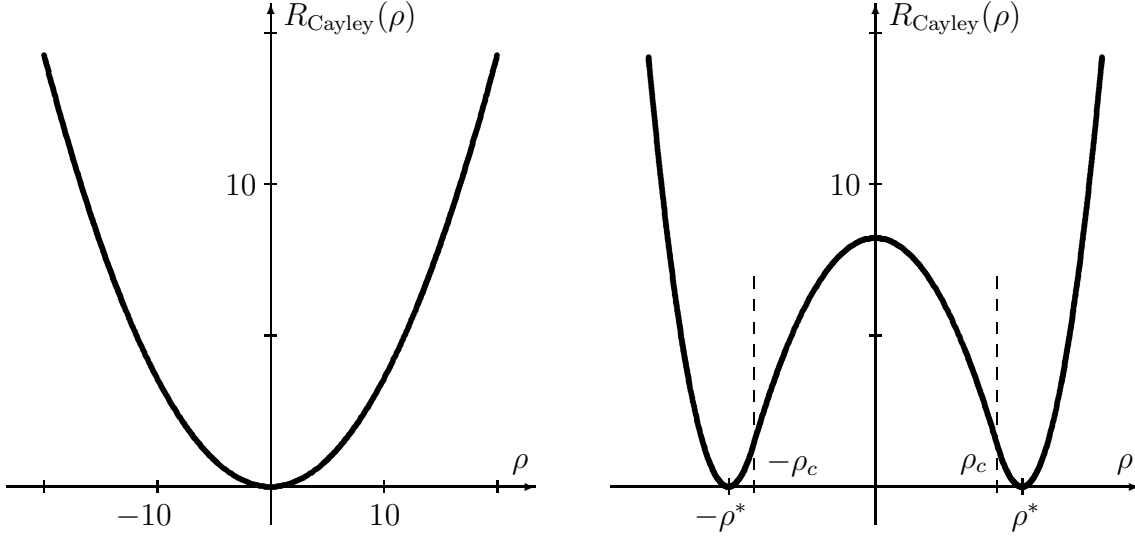


Figure 3: Rate functions for the order parameter  $r_{N,\frac{3}{2}}$  of the spherical model on a Cayley tree for  $\beta = \beta_c$  (left) and for  $\beta = 2\beta_c$  (right).

On applying the saddle-point method to the integral (32) and using Eq. (24) for the main asymptotics of the partition function we obtain the following expression for the distribution density of the order parameter

$$f_{N,\frac{3}{2}}(\rho) = \exp\left(-Nn^{-3}R_{\text{Cayley}}(\rho) + O(N/n^5)\right), \quad (35)$$

where

$$R_{\text{Cayley}}(\rho) = \sqrt{2}\beta J \times \begin{cases} \left[ \frac{\pi}{\sqrt{2}-1} \sqrt{1 - \frac{\beta_c}{\beta}} - \frac{|\rho|}{2}(\sqrt{2}-1) \right]^2, & \text{if } |\rho| \geq \rho_c; \\ \pi^2 \left(1 - \frac{\beta_c}{\beta}\right) - \frac{\rho^2}{8}(\sqrt{2}-1)^3, & \text{if } |\rho| < \rho_c. \end{cases}$$

The main features of the rate function  $R_{\text{Cayley}}(\rho)$  are shown in Figure 3. It vanishes at the points

$$\rho = \pm\rho^* \equiv \pm \frac{2\pi}{(\sqrt{2}-1)^2} \sqrt{1 - \frac{\beta_c}{\beta}}.$$

These points are the equilibrium values of the order parameter  $r_{N,\frac{3}{2}}$  of the spherical model on a binary Cayley tree.

The critical level  $\rho_c$ , given by Eq. (34), can be also interpreted as a relationship for the critical temperature of the “spherical” lattice gas on a Cayley tree. If we consider the ensemble with the gas density

$$\frac{1}{N} \sum_{(j,k) \in T_n} x_{j,k}$$

fixed at  $n^{-3/2}\rho$ , then, as it follows from Eq. (34), a phase separation takes place in the system when the temperature drops below

$$T_{\text{ph.s}} = T_c \left[ 1 - \left( \frac{\sqrt{2}-1}{4\pi} \right)^2 \rho^2 \right]. \quad (36)$$

Since the saddle-point of the integral (32) sticks at the second-largest eigenvalue  $\tau_{n-1,1}$ , the formula for the eigenvector  $\mathbf{v}^{(n-1,1)}$  suggest the following picture below the phase-separation temperature  $T_{\text{ph.s}}$ . The high-density phase ( $x_{j,k} > 0$ ) gathers in one half of the binary tree, while the low-density phase ( $x_{j,k} < 0$ ) is all that remains in the other half of the binary tree.

## 8 Discussion and concluding remarks.

The analysis of previous sections can be extended to the case of a Cayley tree with branching ratio  $q > 2$  at the expense of extra technical efforts. According to [2], the second-largest eigenvalue  $\tau_{n-1,1}$  is  $(q-1)$ -times degenerate. This degeneracy has certain macroscopically observable consequences in lattice-gas models, that is, in models with fixed value of properly normalized gas density (total spin).

Phase separation in lattice-gas spherical models is a condensation into eigenvectors corresponding to the second-largest eigenvalue  $\tau_{n-1,1}$ . In the case  $q = 2$  the condensation scenario is quite simple. An  $n$ -generation Cayley tree,  $T_n$ , consists from the root, the left sub-tree  $T_{n-1}^L$  and the right subtree  $T_{n-1}^R$ , see Fig. 4. Accordingly, the eigenvector  $\mathbf{v}^{(n-1,1)}$  is a combination of the high-density and low-density ground states in the left and right subtrees (or vice-versa) and zero-value at the root. Therefore, the condensation scenario on a binary tree is quite simple. If the fixed value of normalized gas density is not equal to the equilibrium value, then the excess of the high-density phase gathers in the left or right subtree, while the low-density phase gathers in the opposite subtree.

If we consider a Cayley tree with branching ratio  $q > 2$ , then the number of phase-separation scenarios increases. For instance, if  $q = 3$ , then  $T_n$  contains three  $(n-1)$ -generation subtrees, and the high-density phase could gather either in one or in two of the three available subtrees.

The widely known solution of the Ising model on a Cayley tree (the IC model) described in the book by Baxter [1] is based on calculating the magnetization induced by a homogeneous field  $h$  applied at the boundary. If the temperature is sufficiently high, then (in the thermodynamic limit) the boundary field has no influence on the random variables  $x_{j,k}$  located close to the root of the tree. However, when the temperature is below  $T_B$ :  $\tanh(J\beta_B) = 1/2$ , an arbitrarily weak boundary field  $h$  induces non-zero expected values of all random variables, including those located around the root of the tree. Namely, the boundary field induces a magnetization  $m_N(h)$  in the middle of the tree, and  $m_N(h)$  converges to a non-zero limit  $m(T) \text{sgn}(h)$  as  $N \rightarrow \infty$ . Moreover,  $m(T)$  does not depends on  $h$ . The penetration temperature  $T_B$  was interpreted in [1] as the critical temperature of the IC model.

As a rule, the penetration temperature and the critical temperature coincide in finite-dimensional systems. However, the situation becomes very different when we consider models on Cayley trees. Indeed, the exact solution reported in the present paper shows that the critical and the penetration temperatures differ in the case of the spherical model. For the characteristic function

$$\chi_{j,k}(t) = \langle \exp(itx_{j,k}) \rangle_n$$

of the random variable  $x_{j,k}$  at the node  $(j, k) \in T_n$  we obtain the following large- $n$  asymptotics

$$\chi_{j,k}(t) \sim \exp \left[ -\frac{t^2}{4\beta J} \sum_{(l,m) \in T_n} \frac{(v_{j,k}^{(l,m)})^2}{z_n^* - \lambda_{l,m}} + \frac{it}{2J} \sum_{(l,m) \in T_n} \frac{\phi_{l,m} v_{j,k}^{(l,m)}}{z_n^* - \lambda_{l,m}} \right], \quad (37)$$

where  $\phi_{l,m}$  are the scalar products of the eigenvectors  $\mathbf{v}^{(l,m)}$  and the homogeneous boundary field, see Eq. (13). Since  $\phi_{l,m} \neq 0$  only for  $l = n$ , the large- $n$  asymptotics of the expected values of the random variables  $x_{j,k}$  are given by

$$\langle x_{j,k} \rangle_n = \frac{1}{2J} \sum_{(l,m) \in T_n} \frac{\phi_{l,m} v_{j,k}^{(l,m)}}{z_n^* - \lambda_{l,m}} = \frac{2^{(n-j)/2} h}{2(n+1)J} \sum_{m=1}^n \frac{2 \sin \frac{\pi n m}{n+1} \sin \frac{\pi j m}{n+1}}{z_n^* - \sqrt{2} \cos \frac{\pi m}{n+1}}.$$

Equation (12) yields

$$\langle x_{j,k} \rangle_n = \frac{2^{(n-j+1)/2} h}{2J} \frac{x_+^j(z_n^*) - x_-^j(z_n^*)}{x_+^{n+1}(z_n^*) - x_-^{n+1}(z_n^*)}.$$

Therefore, the effective field generated by the boundary conditions penetrates inside the tree once  $x_+(z^*) \leq \sqrt{2}$ , that is, once  $z^* \leq \frac{3}{2}$ . Looking at the saddle-point equation (21) we conclude that the penetration temperature of the spherical model is given by

$$T_p = \frac{J}{W_{\text{Cayley}}(3/2)} \left( 1 - \frac{1}{\sqrt{2}} \left( \frac{h}{2J} \right)^2 \right),$$

where

$$W_{\text{Cayley}}(3/2) = \sum_{j=2}^{\infty} 2^{-j} \left( j \frac{2^j + 1}{2^j - 1} - 3 \right).$$

The penetration temperature  $T_p$  is the direct analogue of the critical temperature of the Ising model on a Bethe lattice,  $T_B$ .

The large-deviation probabilities for the order-parameter  $r_{n, \frac{3}{2}}$  decay exponentially with  $N/n^3$  (not with  $N$ ). Nevertheless, the low-temperature phases of the spherical model on a Cayley tree should be classified as rigid. Indeed, the benchmark of rigidity is the behavior of large-deviation probabilities for  $T > T_c$ . According to Eqs. (33) and (35) large-deviation probabilities for  $r_{n, \frac{3}{2}}$  decay exponentially with  $N/n^3$  both below and above the critical temperature  $T_c$ . Therefore, the low-temperature phases are rigid.

## Appendix A. Spectral Properties of Cayley-Tree Matrices.

Often, methods developed for solving various 1D models are successfully applied to the corresponding models on Cayley trees. Not surprisingly, after a minor effort, calculation of eigenvalues  $\lambda_{k,l}$  and eigenvectors  $\mathbf{u}^{(k,l)}$  of the Cayley-tree matrix  $\widehat{M}_N$  having  $N = 2^n - 1$  rows and columns, see Eq. (2), is reduced to investigation of spectral properties of tri-diagonal matrices.

We use the symbols  $\lambda_{k,l}$  and  $\mathbf{u}^{(k,l)}$  to denote an abstract complete set of  $N$  eigenvalues and orthonormal eigenvectors of the Cayley-tree matrix  $\widehat{M}_N$ . For instance, we use these notations to diagonalise the Hamiltonian  $H_n$  of the spherical model, see Eq. (1). In this case, the ranges of indexes  $k$  and  $l$  in the eigenvalues  $\lambda_{k,l}$  and eigenvectors  $\mathbf{u}^{(k,l)}$  mimic the labelling of nodes in the tree  $T_n$ :  $l = 1, 2, \dots, 2^{k-1}$ ; and  $k = 1, 2, \dots, n$ .

We use the symbols  $\tau_{k,l}$  and  $\mathbf{v}^{(k,l;j)}$  to denote the spectrum (the set of different eigenvalues) of the matrix  $\widehat{M}_N$  and the corresponding eigenvectors. In this case, as we shall see below, the indexes  $k$  and  $l$  run over the triangular array  $l = 1, 2, \dots, k$ ;  $k = 1, 2, \dots, n$ . The multiplicities of eigenvalues  $\tau_{k,l}$  are denoted either  $m_{n;k,l}$ , or  $m_{k,l}$  if it is clear from the context how many generation the Cayley tree contains. The last index in  $\mathbf{v}^{(k,l;j)}$ , separated from the others by a semicolon, reflects the multiplicity of eigenvalue  $\tau_{k,l}$ :  $j = 1, 2, \dots, m_{k,l}$ . If  $m_{k,l} = 1$ , then one can omit the multiplicity index  $j$ ,  $\mathbf{v}^{(k,l;j)} = \mathbf{v}^{(k,l;1)} = \mathbf{v}^{(k,l)}$ . In the recursive procedure used below for constructing the complete set of eigenvectors of  $\widehat{M}_N$  it might be necessary to specify explicitly the number of tree generations,  $n$ . Alas, in such cases we have to append one more superscript to eigenvectors and use symbols like  $\mathbf{v}^{(n;k,l;j)}$  or  $\mathbf{v}^{(n;k,l)}$ . If specified at all, the number of generations,  $n$ , is always the first single index separated from the others by a semicolon.

We use two kinds of indexing for the components of a vector  $\mathbf{x}$ . When the components must be ordered explicitly we use a single index,  $\mathbf{x} = \{x_j : j = 1, 2, \dots, 2^n - 1\}$ . If it is necessary to take into account the tree structure, we mimic the indexing of nodes in the tree  $T_n$ :  $\mathbf{x} = \{x_{i,j} : j = 1, 2, \dots, 2^{i-1}; i = 1, 2, \dots, n\}$ . The component  $x_j$  with  $j = 2^{n-1}$  of an explicitly ordered vector  $\mathbf{x}$  always corresponds to the root (the node (1, 1)) of the tree  $T_n$ . Therefore, we call  $x_j$  with  $j = 2^{n-1}$  the root component.

We begin our quest for spectral properties of Cayley-tree matrix  $\widehat{M}_N$  with finding all special eigenvectors  $\mathbf{v} \equiv \{v_{j,k} : (j, k) \in T_n\}$  having the following form

$$v_{j,1} = v_{j,2} = \dots = v_{j,2^{j-1}} = y_j, \quad \text{for } j = 1, 2, \dots, n. \quad (38)$$

That is, our first aim is to find all eigenvectors with identical components along each generation of the tree. Since, the components  $v_{j,k}$  of any eigenvector satisfy a linear relationship of the form

$$\frac{1}{2}v_{j-1,l} + \frac{1}{2}v_{j+1,m} + \frac{1}{2}v_{j+1,m+1} = \lambda v_{j,k},$$

the components of the vector  $\mathbf{y} \equiv \{y_j\}_{j=1}^n$  satisfy the relationship  $\frac{1}{2}y_{j-1} + y_{j+1} = \lambda y_j$ .

Hence, the vector  $\mathbf{y}$  is one of the eigenvectors of the  $n \times n$  tri-diagonal matrix

$$\hat{L}_n = \begin{pmatrix} 0 & 1 & & & & \\ \frac{1}{2} & 0 & 1 & & & 0 \\ & \frac{1}{2} & 0 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & 0 & 1 \\ 0 & & & & \frac{1}{2} & 0 & 1 \\ & & & & & \frac{1}{2} & 0 \end{pmatrix}.$$

Therefore (non-degenerate) eigenvalues  $\tau_{n,l}$  corresponding to special eigenvectors  $\mathbf{v}^{(n,l)}$  of the matrix  $\widehat{M}_N$  coincide with the eigenvalues  $\Lambda_{n,l} = \sqrt{2} \cos \frac{\pi l}{n+1}$ ,  $l = 1, 2, \dots, n$  of the matrix  $\hat{L}_n$ . The eigenvectors of  $\hat{L}_n$  are given by

$$\mathbf{y}^{(n;l)} = \{y_k^{(n;l)}\}_{k=1}^n = \left\{ 2^{-k/2} \sin \frac{\pi k l}{n+1} \right\}_{k=1}^n, \quad k = 1, 2, \dots, n.$$

Note that the first component  $y_1^{(n;l)}$  of any eigenvector  $\mathbf{y}^{(n;l)}$  is greater than 0.

Since the matrix  $\hat{L}_n$  is asymmetric, the vectors  $\mathbf{y}^{(n;l)}$  are linearly independent but they are not orthogonal. Formally, the reason for the lack of orthogonality are the multipliers  $2^{-k/2}$ . However, there are exactly  $2^{k-1}$  nodes  $(k, j)$  in the  $k$ -th generation of a binary tree  $T_n$ . Therefore, the special eigenvectors  $\mathbf{v}^{(n,l)}$  constructed from the vectors  $\mathbf{y}^{(n;l)}$  according to Eq. (38) are orthogonal.

Complete sets of eigenvalues and eigenvectors for the matrices

$$\widehat{M}_N \quad \text{with} \quad N = 2^n - 1, \quad n = 2, 3, \dots$$

can be constructed from the eigenvalues and eigenvectors of  $\widehat{M}_{(N-1)/2}$  and from the eigenvalues  $\Lambda_{n,l} (= \tau_{n,l})$  of the matrix  $\hat{L}_n$  and the corresponding special eigenvectors  $\mathbf{v}^{(n,l)}$ . In the case of a tree with two generations,  $T_2$ , the eigenvalues and eigenvectors of the matrix  $\hat{L}_2$  are given by

$$\Lambda_{2;1} = -\frac{1}{\sqrt{2}}, \quad \Lambda_{2;2} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \mathbf{y}^{(2;1)} = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}, \quad \mathbf{y}^{(2;2)} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}.$$

The corresponding special eigenvectors of the Cayley-tree matrix

$$\widehat{M}_3 = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad \text{are given by} \quad \mathbf{v}^{(2,1)} = \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}, \quad \mathbf{v}^{(2,2)} = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}. \quad (39)$$

The remaining eigenvalue of the matrix  $\widehat{M}_3$  is  $\tau_{1,1} = 0$ , and the corresponding eigenvector is  $\mathbf{v}^{(1,1)} = (1, 0, -1)^T$ .

Before turning to the general induction step it is instructive to see how one can construct eigenvalues and eigenvectors of the Cayley-tree matrix

$$\widehat{M}_7 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & & & & \\ 0 & 0 & \frac{1}{2} & & & & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & & & \\ & & \frac{1}{2} & 0 & \frac{1}{2} & & \\ & & & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & & & & \frac{1}{2} & 0 & 0 \\ & & & & & \frac{1}{2} & 0 & 0 \end{pmatrix},$$

from the eigenvalues and eigenvectors of the matrix  $\widehat{M}_3$ .

The last three eigenvalues of  $\widehat{M}_7$  are the eigenvalues of the matrix  $\widehat{L}_3$ :

$$\tau_{3,1} = \Lambda_{3,1} = -1, \quad \tau_{3,2} = \Lambda_{3,2} = 0, \quad \tau_{3,3} = \Lambda_{3,3} = 1.$$

The eigenvectors of the matrix  $\widehat{L}_3$  are given by

$$\mathbf{y}^{(3;1)} = \begin{pmatrix} 1 \\ -1 \\ \frac{1}{2} \end{pmatrix}, \quad \mathbf{y}^{(3;2)} = \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \quad \mathbf{y}^{(3;3)} = \begin{pmatrix} 1 \\ 1 \\ \frac{1}{2} \end{pmatrix}.$$

Hence, the special (constant along generation) eigenvectors of the matrix  $\widehat{M}_7$  are

$$\begin{aligned} \mathbf{v}^{(3,1)} &= \left( \frac{1}{2}, \frac{1}{2}, -1, 1, -1, \frac{1}{2}, \frac{1}{2} \right)^T, \\ \mathbf{v}^{(3,2)} &= \left( -\frac{1}{2}, -\frac{1}{2}, 0, 1, 0, -\frac{1}{2}, -\frac{1}{2} \right)^T, \\ \mathbf{v}^{(3,3)} &= \left( \frac{1}{2}, \frac{1}{2}, 1, 1, 1, \frac{1}{2}, \frac{1}{2} \right)^T. \end{aligned}$$

The remaining eigenvalues  $\tau_{1,1}$ ,  $\tau_{2,1}$ ,  $\tau_{2,2}$  of  $\widehat{M}_7$  are identical to the eigenvalues  $\tau_{k,l}$  of the matrix  $\widehat{M}_3$ , but, as we will see shortly, the eigenvalue  $\tau_{1,1}$  of  $\widehat{M}_7$  is twice degenerate,  $m_{3,1,1} = 2$ .

To find the eigenvectors of the matrix  $\widehat{M}_7$  corresponding to  $\tau_{1,1}$ , take the eigenvector  $\mathbf{v}^{(2;1,1)}$  of  $\widehat{M}_3$  and note that the root component of  $\mathbf{v}^{(2;1,1)}$  is zero,  $v_2^{(2;1,1)} = 0$ . The vectors  $(1, -1, 0)^T$  and  $(0, -1, 1)^T$  — permutations of  $\mathbf{x}^{(2;1,1)}$  — are eigenvectors of the  $3 \times 3$  blocks in, respectively, the upper left and lower right corners of  $\widehat{M}_7$ . Therefore the vectors

$$\mathbf{v}^{(3;1,1;1)} = (1, -1, 0, 0, 0, 0, 0)^T \quad \text{and} \quad \mathbf{v}^{(3;1,1;2)} = (0, 0, 0, 0, 0, -1, 1)^T$$

are eigenvectors of the matrix  $\widehat{M}_7$  corresponding to the eigenvalue  $\tau_{1,1}$ . Note that we are able to construct two orthogonal eigenvectors of the matrix  $\widehat{M}_7$  from the eigenvector  $\mathbf{v}^{(2;1,1)}$ , because the root component of  $\mathbf{v}^{(2;1,1)}$  is equal to 0.

The remaining eigenvectors of  $\widehat{M}_7$  are constructed from the eigenvectors  $\mathbf{v}^{(2;2,1)}$  and  $\mathbf{v}^{(2;2,2)}$  of the matrix  $\widehat{M}_3$ , see Eq. (39).

The permutations

$$\begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \sqrt{2} \\ 1 \\ 1 \end{pmatrix}$$

of  $\mathbf{v}^{(2;2,2)}$  are eigenvectors of, respectively, the upper left and lower right  $3 \times 3$  blocks of the matrix  $\widehat{M}_7$ . An inspection shows that the anti-symmetric vector

$$\mathbf{v}^{(3;2,2)} = (1, 1, \sqrt{2}, 0, -\sqrt{2}, -1, -1)^T$$

is the eigenvector of the matrix  $\widehat{M}_7$  corresponding to the eigenvalue  $\lambda_{2,2} = \frac{1}{\sqrt{2}}$ . Analogously, the anti-symmetric vector

$$\mathbf{v}^{(3;2,1)} = (-1, -1, \sqrt{2}, 0, -\sqrt{2}, 1, 1)^T$$

constructed from permutations of  $\mathbf{v}^{(2;2,1)}$  is the eigenvector of the matrix  $\widehat{M}_7$  corresponding to the eigenvalue  $\lambda_{2,1} = -\frac{1}{\sqrt{2}}$ . Note that the root components of the eigenvectors  $\mathbf{v}^{(2;2,1)}$  and  $\mathbf{v}^{(2;2,2)}$  are not equal to zero and each of those vectors generates exactly one eigenvector of the matrix  $\widehat{M}_7$ . The three special eigenvectors together with the four eigenvectors constructed from the eigenvectors of the matrix  $\widehat{M}_3$  make up a complete set of eigenvectors of the matrix  $\widehat{M}_7$ .

Using essentially the same procedure one can construct the eigenvalues and eigenvectors of a matrix  $\widehat{M}_{2N+1}$  from those of the matrix  $\widehat{M}_N$ , where  $N = 2^n - 1$ . The first  $n(n+1)/2$  eigenvalues  $\tau_{k,l}$  of  $\widehat{M}_{2N+1}$  are the eigenvalues of  $\widehat{M}_N$ . The remaining  $n+1$  eigenvalues of  $\widehat{M}_{2N+1}$  are the eigenvalues  $\tau_{n+1,l} = \Lambda_{n+1,l}$  of the matrix  $\widehat{L}_{n+1}$ . A complete set of  $n+1$  eigenvectors of  $\widehat{L}_{n+1}$  generates  $n+1$  special (constant along generations) orthogonal eigenvectors of the matrix  $\widehat{M}_{2N+1}$ . Each eigenvector of  $\widehat{M}_N$  with the root component equal to 0 generates a pair of orthogonal eigenvectors of  $\widehat{M}_{2N+1}$  (with zero root components). Finally, the set of  $n$  special eigenvectors of  $\widehat{M}_N$  generates  $n$  orthogonal anti-symmetric eigenvectors of  $\widehat{M}_{2N+1}$  (with the root components equal to zero). Note that according to the above construction only special eigenvectors have non-zero root components.

To justify the above recursive procedure we construct the matrix  $\widehat{M}_{2N+1}$  from a pair of matrices  $\widehat{M}_N$  used as building blocks. We begin from arranging in a simple sequence the labels  $(j, k)$  of the matrix elements  $M_{(j,k),(l,m)}$ . That is, we number the labels  $(j, k)$  by integers  $1, 2, \dots, 2N+1$ . Denote  $\widehat{U}_N$  the matrix obtained from  $\widehat{M}_N$  if we number the labels from the bottom of a tree to the top. The exact numbering algorithm is not important, but the root  $(1, 1)$  must receive the highest number  $N \equiv 2^n - 1$ . Denote  $\widehat{D}_N$  the matrix obtained by numbering the labels  $(j, k)$  of  $\widehat{M}_N$  in the opposite order, from top to bottom. Now  $M_{(1,1),(k,l)}$  are the elements of the first row of the matrix  $\widehat{M}_N$ . Note that, if  $(v_1, v_2, \dots, v_N)$  is an eigenvector of  $\widehat{U}_N$ , then  $(v_N, v_{N-1}, \dots, v_1)$  is an eigenvector of  $\widehat{D}_N$  with the same eigenvalue.



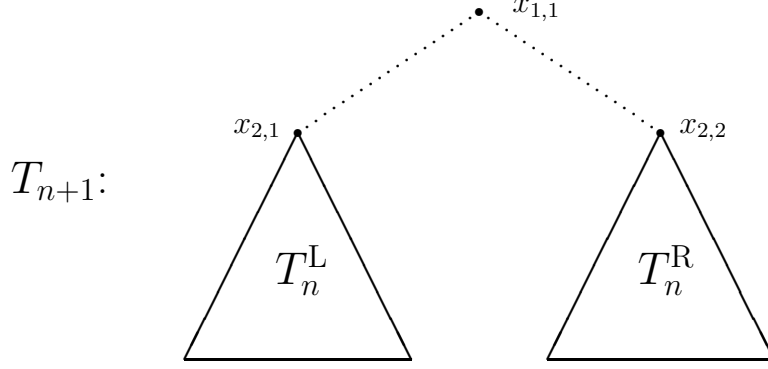


Figure 4: An  $(n + 1)$ -generation Cayley tree,  $T_{n+1}$ , consists from the root, with the attached variable  $x_{1,1}$ , and from two identical  $n$ -generation trees,  $T_n^L$  and  $T_n^R$  (left and right), with the variables  $x_{2,1}$  and  $x_{2,2}$  attached to their roots.

Using the two matrices  $\widehat{U}_N$  and  $\widehat{D}_N$  as building blocks we can construct the Cayley-tree matrix  $M_{2N+1}$  as shown in Fig. 4. We number the labels of the left subtree by integers  $1, 2, \dots, 2^n - 1$  from bottom to top, assign the number  $2^n$  to the root of  $T_{n+1}$ , and number the right subtree by integers  $2^n + 1, 2^n + 2, \dots, 2^{n+1} - 1$  from top to bottom in the opposite order. The obtained Cayley-tree matrix  $\widehat{M}_{2N+1}$  looks like this

$$\widehat{M}_{2N+1} = \begin{pmatrix} u_{1,1} & \dots & u_{1,N} & & & & \\ \vdots & \ddots & \vdots & & & & \\ u_{N,1} & \dots & u_{N,N} & \frac{1}{2} & & & \\ & & \frac{1}{2} & 0 & \frac{1}{2} & & \\ & & & \frac{1}{2} & d_{1,1} & \dots & d_{1,N} \\ & 0 & & & \vdots & \ddots & \vdots \\ & & & & d_{N,1} & \dots & d_{N,N} \end{pmatrix}.$$

If we already know the eigenvalues and eigenvectors of the matrices  $\widehat{U}_N$  and  $\widehat{D}_N$ , then the eigenvectors of the matrix  $\widehat{M}_{2N+1}$  are constructed as follows. If  $\mathbf{v}$  is an eigenvector of  $\widehat{U}_N$  with eigenvalue  $\tau$  and with zero root component, that is, if  $v_N = 0$ , then  $(v_1, \dots, v_N, 0, \dots, 0)$  is an eigenvector of  $\widehat{M}_{2N+1}$  with the same eigenvalue  $\tau$ . Due to the simple relationship between the eigenvectors of  $\widehat{U}_N$  and  $\widehat{D}_N$  mentioned above, the vector  $(0, \dots, 0, v_N, \dots, v_1)$  is also an eigenvector of  $\widehat{M}_{2N+1}$  with the eigenvalue  $\tau$ .

Let now  $\mathbf{v}^{(n;n,l)}$  be a special eigenvector of  $\widehat{U}_N$  with the eigenvalue  $\tau_{n,l}$ , then

$$\mathbf{v}^{(n+1;n,l)} \equiv (v_1, \dots, v_N, 0, -v_N, \dots, -v_1)$$

is an eigenvector of  $\widehat{M}_{2N+1}$  with the same eigenvalue. Note that  $\mathbf{v}^{(n+1;n,l)}$  is an anti-symmetric vector with zero root component.

Now we are going to show that the obtained eigenvectors with zero-root components and the  $n + 1$  special eigenvectors obtained from linearly independent eigenvectors of the matrix  $\widehat{L}_{n+1}$  comprise the complete set of orthogonal eigenvectors of the matrix  $\widehat{M}_{2N+1}$ . Since the constructed eigenvectors are orthogonal, the obtained set is complete if it contains  $2N + 1$  vectors. According to our recursive procedure, the matrix  $\widehat{M}_N$  has  $n$  special eigenvectors and  $N - n$  eigenvectors with zero root component. Therefore, the matrix  $\widehat{M}_{2N+1}$ , has exactly  $2(N - n) + n = 2N - n$  eigenvectors with zero root component. Together with  $n + 1$  special eigenvectors we obtain  $2N + 1$  orthogonal vectors. Therefore the obtained set is the complete set of eigenvectors of the matrix  $\widehat{M}_{2N+1}$ .

## Appendix B. Useful Sums and their Asymptotics.

In this appendix we derive an explicit expression for the sum

$$\Sigma_n(z) = \sum_{(j,k) \in T_n} \frac{\gamma_{j,k}^2}{z - \lambda_{j,k}},$$

see Eq. (28), but we begin with a formula for the coefficients

$$\gamma_{j,k} \equiv \sum_{(l,m) \in T_n} u_{l,m}^{(j,k)}.$$

By construction, see Appendix A, only the special (constant along generations) eigenvectors  $\mathbf{v}^{(n,k)}$  have non-zero sums of their components. Therefore

$$\Sigma_n(z) = \sum_{k=1}^n \frac{\delta_{n,k}^2}{z - \tau_{n,k}}, \quad \text{where} \quad \delta_{n,k} = \sum_{(l,m) \in T_n} v_{l,m}^{(n,k)}.$$

The normalized special eigenvectors  $\mathbf{v}^{(n,k)}$ ,  $k = 1, 2, \dots, n$ , are given by

$$\left\{ v_{l,m}^{(n,k)} \right\}_{(l,m) \in T_n} = \left\{ \frac{2^{1-l/2}}{\sqrt{n+1}} \sin \frac{\pi l k}{n+1} \right\}_{(l,m) \in T_n}.$$

Hence

$$\begin{aligned} \delta_{n,k} &= \sum_{l=1}^n \sum_{m=1}^{2^{l-1}} \frac{2^{1-l/2}}{\sqrt{n+1}} \sin \frac{\pi l k}{n+1} = \frac{1}{\sqrt{n+1}} \sum_{l=1}^n 2^{l/2} \sin \frac{\pi l k}{n+1} \\ &= \frac{1}{\sqrt{n+1}} \operatorname{Im} \sum_{l=1}^n \left( \sqrt{2} \exp \frac{i \pi k}{n+1} \right)^l = \sqrt{\frac{2}{n+1}} \sin \frac{\pi k}{n+1} \frac{1 - (-1)^k 2^{\frac{n+1}{2}}}{3 - 2\sqrt{2} \cos \frac{\pi k}{n+1}}. \end{aligned}$$

Thus

$$\Sigma_n(z) = \frac{2}{n+1} \sum_{k=1}^n \frac{\sin^2 \frac{\pi k}{n+1}}{(3 - 2\sqrt{2} \cos \frac{\pi k}{n+1})^2} \frac{1 + 2^{n+1} - (-1)^k 2^{\frac{n+3}{2}}}{z - \sqrt{2} \cos \frac{\pi k}{n+1}}.$$

To find a manageable expression for  $\Sigma(z)$  let us introduce an extra parameter  $a$  and consider the sum

$$S(a, z) = \frac{2}{n+1} \sum_{k=1}^n \frac{\sin^2 \frac{\pi k}{n+1}}{(a - 2\sqrt{2} \cos \frac{\pi k}{n+1})^2} \frac{1}{z - \sqrt{2} \cos \frac{\pi k}{n+1}}.$$

Now one can decimate the degree of the denominator and split the sum in two parts

$$\begin{aligned} S(a, z) &= -\frac{2}{n+1} \frac{d}{da} \sum_{k=1}^n \frac{\sin^2 \frac{\pi k}{n+1}}{a - 2\sqrt{2} \cos \frac{\pi k}{n+1}} \frac{1}{z - \sqrt{2} \cos \frac{\pi k}{n+1}} \\ &= -\frac{1}{n+1} \frac{d}{da} \frac{1}{z - a/2} \left( \sum_{k=1}^n \frac{\sin^2 \frac{\pi k}{n+1}}{a/2 - \sqrt{2} \cos \frac{\pi k}{n+1}} - \sum_{k=1}^n \frac{\sin^2 \frac{\pi k}{n+1}}{z - \sqrt{2} \cos \frac{\pi k}{n+1}} \right). \end{aligned}$$

Using the formula (12) with  $j = n$  we obtain

$$S(a, z) = \frac{2^{-3/2}}{\left(z - \frac{a}{2}\right)^2} \left[ \frac{x_+^{2n+1}(z) - x_+(z)}{x_+^{2(n+1)}(z) - 1} - \frac{x_+^{2n+1}(a/2) - x_+(a/2)}{x_+^{2(n+1)}(a/2) - 1} \right] \\ + \frac{2^{-1/2}}{z - \frac{a}{2}} \left[ 2(n+1) \frac{x_+^{4n+2}(a/2) - x_+^{2n+2}(a/2)}{\left(x_+^{2(n+1)}(a/2) - 1\right)^2} - \frac{(2n+1)x_+^{2n}(a/2) - 1}{x_+^{2(n+1)}(a/2) - 1} \right] \frac{x_+(a/2)}{\sqrt{a^2 - 8}}.$$

In particular

$$S(3, z) = \frac{2^{-3/2}}{\left(z - \frac{3}{2}\right)^2} \left[ \frac{x_+^{2n+1}(z) - x_+(z)}{x_+^{2(n+1)}(z) - 1} - \frac{2^{n+1/2} - \sqrt{2}}{2^{n+1} - 1} \right] + \\ + \frac{1}{z - \frac{3}{2}} \frac{2^{2n+1} - n2^{n+1} - 2^n - 1}{(2^{n+1} - 1)^2}. \quad (40)$$

Therefore for any  $z > \sqrt{2}$  we obtain the following large- $n$  asymptotics

$$\frac{N}{\Sigma_n(z)} = \frac{2\left(z - \frac{3}{2}\right)^2}{3z - 4 - \sqrt{z^2 - 2}} + O\left[x_+^{-2n}(z) + n2^{-n}\right].$$

If we introduce a new variable  $\zeta$  via  $z = \sqrt{2}(1 + \zeta n^{-2})$ , then we obtain the following asymptotics

$$\frac{N}{\Sigma_n(\sqrt{2}(1 + \zeta n^{-2}))} = \frac{\left(\sqrt{2}(1 + \zeta n^{-2}) - \frac{3}{2}\right)^2}{\frac{1}{\sqrt{2}} \frac{x_+^{2n+1}(1+\zeta n^{-2}) - x_+(1+\zeta n^{-2})}{x_+^{2n+2}(1+\zeta n^{-2}) - 1} + \sqrt{2}(1 + \zeta n^{-2}) - 2} + O\left(n2^{-n}\right).$$

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